Solving a Log-Linearized RBC Model Algebraically

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These notes describe how to construct a log-linear approximation of an RBC model around a non-stochastic steady state and how to solve the linearized model algebraically using the method of undetermined coefficients. The procedure described below is used to produce the simulations on this webpage: https://www.briancjenkins.com/simulations/centralized-rbc.html. The exercise is tedious and demonstrates the great value provided by the many computational tools and methods available for approximating and solving DSGE models.

The Model The equilibrium conditions of the model are given by:

$$\varphi (1 - L_t)^{-\eta} = C_t^{-\sigma} W_t \tag{1}$$
$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-\sigma} R_{t+1} + 1 - \delta) \right] \tag{2}$$
$$W_t = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} \tag{3}$$
$$R_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} \tag{4}$$
$$K_{t+1} = I_t + (1-\delta) K_t \tag{5}$$
$$Y_t = C_t + I_t \tag{6}$$
$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{7}$$
$$\log A_{t+1} = \rho_a \log A_t + \epsilon_{t+1} \tag{8}$$

Equation (1) is the representative household's first-order condition for the optional choice of labor. Equation (2) is the household's Euler equation reflecting an optimal choice of capital for period $t+1$. Equations (3) and (4) are the firm sector's first-order conditions for optimal choices of labor and capital. Equations (5), (6), and (7) describe the evolution of the aggregate capital stock, the goods market clearing condition, and the production function. Finally, equation (8) indicates that log TFP follows an AR(1) process where $\epsilon_{t+1} \sim \mathcal{N}\left(0, \sigma_\epsilon^2\right)$ is an exogenous shock process.

Steady State Let capital letters without time subscripts denote non-stochastic steady state values of the model variables. The steady state is given by the following system:

$$\varphi (1 - L)^{-\eta} = C^{-\sigma} W \tag{9}$$
$$C^{-\sigma} = \beta E \left[C^{-\sigma} (R + 1 - \delta) \right] \tag{10}$$
$$W = (1-\alpha) A K^{\alpha} L^{-\alpha} \tag{11}$$
$$R = \alpha A K^{\alpha-1} L^{1-\alpha} \tag{12}$$
$$K = I + (1-\delta) K \tag{13}$$
$$Y = C + I \tag{14}$$
$$Y = A K^{\alpha} L^{1-\alpha} \tag{15}$$
$$\log A = \rho_a \log A \tag{16}$$

In general, the nonlinear terms in equation (9) prevent this system from being solved algebraically. However, most of the work can be done with algebra. First, note that steady state TFP is trivial:

$$\boxed{A = 1} \tag{17}$$

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Next, use the Euler and capital demand equations to solve for the steady state capital-to-labor ratio:

$$\frac{K}{L} = \left(\frac{\alpha A}{\beta^{-1} + \delta - 1}\right)^{\frac{1}{1-\alpha}} \tag{18}$$

Next, use equations (9), (11), (13), (14), (15), and (18) to obtain an equation that characterizes steady state labor:

$$\boxed{\varphi(1-L)^{-\eta} L^{\sigma} = (1-\alpha)\left[A(K/L)^{\alpha} - \delta(K/L)\right]^{-\sigma} A(K/L)^{\alpha}} \tag{19}$$

Note that the right side of equation (19) is a constant. Solve (19) numerically.

The remaining steady state values are readily obtained:

$$\begin{aligned}
K &= (K/L)L && (20)\\
Y &= AK^{\alpha}L^{1-\alpha} && (21)\\
W &= (1-\alpha)Y/L && (22)\\
R &= \alpha Y/K && (23)\\
I &= \delta K && (24)\\
C &= Y - I && (25)
\end{aligned}$$

Log-linear Approximation Let lowercase letters represent the log deviations of variables from their steady states, e.g.: $y_t = \log(Y_t) - \log(Y)$. The log-linearized equilibrium conditions are:

$$\begin{aligned}
0 &= -\sigma c_t + w_t - \frac{\eta L}{1-L} l_t && (26)\\
-\sigma E_t c_{t+1} + \beta R E_t r_{t+1} &= -\sigma c_t && (27)\\
0 &= y_t - l_t - w_t && (28)\\
0 &= y_t - k_t - r_t && (29)\\
k_{t+1} &= \delta i_t + (1-\delta) k_t && (30)\\
0 &= \frac{C}{Y} c_t + \frac{I}{Y} i_t - y_t && (31)\\
0 &= a_t + \alpha k_t + (1-\alpha) l_t - y_t && (32)\\
a_{t+1} &= \rho a_t + \epsilon_{t+1} && (33)
\end{aligned}$$

Note that all forward-looking variables have been moved to the left sides of the equations and contemporaneous variables to the left.

Solution The goal is to use algebra to eliminate all endogenous variables from the system except for k_t and to solve the resulting second-order difference equation using the method of undetermined coefficients. To begin, take (26) and use equations (28), (31), and (32) to express labor as a function of only a_t, k_t, and c_t:

$$l_t = \phi_1 a_t + \phi_2 k_t + \phi_3 c_t \tag{34}$$

where:

$$\begin{aligned}
\phi_1 &= \frac{1-L}{\alpha + (\eta - \alpha)L} && (35)\\
\phi_2 &= \alpha \phi_1 && (36)\\
\phi_3 &= -\sigma \phi_1 && (37)
\end{aligned}$$

Then, use the capital evolution equation (30) and the market clearing condition (31) to express consumption as a function of a_t, k_t, and k_{t+1}:

$$c_t = \phi_4 a_t + \phi_5 k_t + \phi_6 k_{t+1} \tag{38}$$

where:

$$\phi_4 = \frac{1 + (1-\alpha)\phi_1}{C/Y - (1-\alpha)\phi_3} \tag{39}$$

$$\phi_5 = \frac{\alpha + (1-\alpha)\phi_2 + \delta^{-1}(1-\delta)I/Y}{C/Y - (1-\alpha)\phi_3} \tag{40}$$

$$\phi_6 = -\frac{\delta^{-1}I/Y}{C/Y - (1-\alpha)\phi_3} \tag{41}$$

Next, use the previous two results to write the real rental rate as a function of only a_t, k_t, and k_{t+1}:

$$r_t = \phi_7 a_t + \phi_8 k_t + \phi_9 k_{t+1} \tag{42}$$

where:

$$\phi_7 = 1 + (1-\alpha)\phi_1 + (1-\alpha)\phi_3\phi_4 \tag{43}$$

$$\phi_8 = \alpha - 1 + (1-\alpha)\phi_2 + (1-\alpha)\phi_3\phi_5 \tag{44}$$

$$\phi_9 = (1-\alpha)\phi_3\phi_6 \tag{45}$$

Now, note that $E_t a_{t+1} = \rho a_t$ and express the Euler equation in terms of only a_t, k_t, k_{t+1}, and $E_t k_{t+2}$:[1]

$$0 = \phi_{10} a_t + \phi_{11} k_t + \phi_{12} k_{t+1} + \phi_{13} E_t k_{t+2} \tag{46}$$

where:

$$\phi_{10} = \sigma\phi_4(1-\rho) + \beta R\phi_7\rho \tag{47}$$

$$\phi_{11} = \sigma\phi_5 \tag{48}$$

$$\phi_{12} = \sigma(\phi_6 - \phi_5) + \beta R\phi_8 \tag{49}$$

$$\phi_{13} = \beta R\phi_9 - \sigma\phi_6 \tag{50}$$

Equation (46) is a second-order difference equation in k_t.

I solve the difference equation in (46) using the method of undetermined coefficients. Guess that the solution takes the following form:

$$k_{t+1} = \pi_1 a_t + \pi_2 k_t \tag{51}$$

where the coefficients π_1 and π_2 are to be found. The guess implies that the expectation k_{t+2} is:

$$E_t k_{t+2} = \pi_1 \rho a_t + \pi_2 k_{t+1} \tag{52}$$

Plug (51) and (52) into (46). π_1 and π_2 are determined by the following system:

$$\phi_{13}\pi_2^2 + \phi_{12}\pi_2 + \phi_{11} = 0 \tag{53}$$

$$\phi_{10} + \phi_{12}\pi_1 + \phi_{13}\pi_1(\rho + \pi_2) = 0 \tag{54}$$

Find π_2 by taking the smallest root of the following quadratic equation:

$$\boxed{\phi_{10} + \phi_{12}\pi_1 + \phi_{13}\pi_1(\rho + \pi_2) = 0} \tag{55}$$

Find π_1 to be:

$$\boxed{\pi_1 = -\frac{\phi_{10}}{\phi_{12} + \phi_{13}(\rho + \pi_2)}} \tag{56}$$

Solutions for c_t, r_t, and l_t:

$$c_t = (\phi_4 + \phi_6\pi_1)\, a_t + (\phi_5 + \phi_6\pi_2)\, k_t \tag{57}$$

$$r_t = (\phi_7 + \phi_9\pi_1)\, a_t + (\phi_8 + \phi_9\pi_2)\, k_t \tag{58}$$

$$l_t = [\phi_1 + \phi_3(\pi_4 + \phi_6\pi_1)]\, a_t + [\phi_2 + \phi_3(\pi_5 + \phi_6\pi_2)]\, k_t \tag{59}$$

[1]Note that a_{t+1} is determined outside of this equation.

Finally, the solutions for the variables i_t, y_t, and w_t are found to be:

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\boxed{
\begin{aligned}
y_t &= [1 + \phi_1 + \phi_3(\pi_4 + \phi_6\pi_1)]\, a_t + [\alpha + \phi_2 + \phi_3(\pi_5 + \phi_6\pi_2)]\, k_t && (60)\\
i_t &= \delta^{-1}\pi_1 a_t + \delta^{-1}(\pi_2 + \delta - 1)k_t && (61)\\
w_t &= [1 - \alpha(\phi_1 + \phi_3\pi_4 + \phi_3\phi_6\pi_1)]\, a_t + \alpha\, [1 - (\phi_2 + \phi_3\pi_5 + \phi_3\phi_6\pi_2)]\, k_t && (62)
\end{aligned}
}
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